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EE 272 - Dynamics of Lasers
Homework 3 : Maxwell-Bloch Equations
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One of the standard forms of single-mode laser equations is derived from the Maxwell–Bloch equations based on the semi-classical theory, where the electric field is described by the Maxwell's equations, the macroscopic atomic polarization is introduced by using Schrödinger equations, and the phenomenological atomic and photon decays are introduced.
The homogeneously broadened single-mode laser equations are described as follows,

$$\frac{dE}{dt} = -\kappa\left[(1+i\Delta)E(t) + AP(t)\right] \tag{1}$$

$$\frac{dP}{dt} = -(1-i\Delta)P(t) - E(t)D(t) \tag{2}$$

$$\frac{dD}{dt} = \gamma\left[1 - D(t) + \frac{1}{2}\left(E^*(t)P(t) + E(t)P^*(t)\right)\right] \tag{3}$$

where $E(t)$ is the electric field (complex variable), $P(t)$ is the atomic polarization (complex variable), and $D(t)$ is the population inversion (real variable). In what follow, the slow envelope component is only considered as the dynamics of electric field in the rate equations. For coherently coupled lasers, the detuning of optical carrier frequencies between two lasers plays an important role such as injection locking. It is thus important to consider the existence of the detuning frequency of the fast optical carrier oscillation of ω_c. In the above equations, time is normalized by the decay rate of the atomic polarization $\gamma_\perp$. $\gamma_\parallel$ is the decay rate of the population inversion ($\gamma = \gamma_\parallel/\gamma_\perp$), and κ_c is the decay rate of the electric field in the laser cavity ($\kappa = \kappa_c/\gamma_\perp$). Δ is the detuning between the optical-carrier frequency ω_c and the atomic resonant frequency ω_a, that is, $\Delta = (\omega_c - \omega_a)/(\kappa_c + \gamma_\perp)$, A is the gain parameter.

Question 1 Given that $E(t)$ and $P(t)$ are treated as having slowly varying amplitude compared with the optical-carrier frequency ω_c, express the complex total electric field and atomic polarization $E_{tot}(t)$, $P_{tot}(t)$.
Assuming $E(t)$ and $P(t)$ have slowly varying amplitudes compared with the optical-carrier frequency ω_c, the complex total electric field and atomic polarization are expressed as,

$$E_{tot}(t) = E(t)e^{-i\omega_c t} + E^*(t)e^{i\omega_c t} \tag{4}$$

$$P_{tot}(t) = P(t)e^{-i\omega_c t} + P^*(t)e^{i\omega_c t} \tag{5}$$

At this stage, it is worthwhile stressing that in case of coherently coupled lasers, the frequency detuning of optical carrier frequencies between the two lasers plays an important role such as injection-locking. It is thus important to consider the existence of the detuning frequency of the fast optical carrier oscillation of ω_c.

Question 2 Assuming that the laser is on resonance, rewrite the Maxwell–Bloch equations considering real parts of $E(t)$, $P(t)$.

Assuming the laser on resonance i.e. $\Delta = 0$ and considering real parts of $E(t)$, $P(t)$, one get,

$$\frac{dE}{dt} = -\kappa\left[E(t) + AP(t)\right] \tag{6}$$

$$\frac{dP}{dt} = -P(t) - E(t)D(t) \tag{7}$$

$$\frac{dD}{dt} = \gamma\left[1 - D(t) + E(t)P(t)\right] \tag{8}$$

Question 3 Find the new set of dynamical rate equations for :
(a) Class C lasers ;
(b) Class B lasers ;
(c) Class A lasers
Explain your methodology in a few sentences. For each case, remind the variables used to describe the dynamics, the conditions (rates), and some examples of lasers.
In order to find the new set of dynamical equations, we have to use the definitions seen in class regarding the adiabatic elimination. Remember that the electric field, the atomic polarization, and the population inversion usually decay on very different time scales, which are given by the decay rates, κ_c(the electric field decay rate), $\gamma_{\|}$ (the population inversion decay rate), and $\gamma_{\perp}$ (the atomic polarization decay rate), respectively. If one of these rates is larger than the others, the corresponding variable relaxes fast and consequently adiabatically adjusts to the other variables. Because the temporal dynamics of the variable with large relaxation rate is faster than the other variables, this variable is regarded as a dependent variable when compared with the other variables. Therefore, the variable with faster relaxation rate is considered to be dependent on the other variables with slower relaxation rates. The number of equations describing the laser can be reduced accordingly.
(a) Class C : $\kappa_c \approx \gamma_{\perp} \approx \gamma_{\|}$
The relaxation rates of the electric field, the population inversion, and the atomic polarization are of the same order. The decay rate of the electric field is comparable to those of the atomic polarization and the population inversion. The dynamics of the three variables are described by Eqs. (6)-(8). Examples of class C lasers are He-Ne (3.39-μm), He-Xe (3.51-μm), and NH_3 lasers
(b) Class B : $\gamma_{\perp} >> \kappa_c >> \gamma_{\|}$
The decay rate of the atomic polarization $\gamma_{\perp}$ is much faster than those of the electric field and the population inversion, and the decay rate of the electric field is faster than that of the population inversion. The atomic polarization dynamics is much faster than the two other variables, and the variable of the atomic polarization is regarded as a dependent variable of the two other variables. In this case, the left-hand-side term of Eq. (7) is considered as zero ($dP/dt = 0$), and Eq. (7) becomes,

$$P(t) = -E(t)D(t) \tag{9}$$

leading to,

$$\frac{dE}{dt} = \kappa\left[-1 + AD(t)\right]E(t) \tag{10}$$

$$\frac{dD}{dt} = \gamma\left[1 - D(t) - E^2(t)D(t)\right] \tag{11}$$

The dynamics of class B lasers are described by the two variables : the electric field $E(t)$ and the population inversion $D(t)$. Many commercial lasers are classified as class B lasers : semiconductor, solid-state, and CO_2 lasers.

(c) Class A : $\gamma_{\perp} \approx \gamma_{||} >> \kappa_c$
The decay rate of the electric field is much slower than those of the atomic polarization and the population inversion. The variables of atomic polarization and the population inversion change much faster than that of the electric field, and they are dependent on the electric field. In this case, the left-hand-side term of Eq. (8) is considered as zero ($dD/dt = 0$), and Eq. (8) becomes,

$$D(t) = \frac{1}{1 + E^2(t)} \tag{12}$$

leading to,

$$\frac{dE}{dt} = \kappa \left[-1 + \frac{A}{1 + E^2(t)} \right] E(t) \tag{13}$$

$$\frac{dE}{dt} \approx \kappa \left[-1 + A - AE^2(t) \right] E(t) \tag{14}$$

Therefore, only one variable of the electric field $E(t)$ is used to describe the dynamics of class A lasers. Examples of class A lasers are He-Ne lasers (632.8-nm) and dye lasers.

Question 4 Which of the aforementioned class(es) can satisfy the conditions for chaos generation ? Explain ?
It has been proven mathematically by the Poincaré–Bendixson theorem that at least three degrees of freedom (i.e., three independent variables) are necessary to observe deterministic chaos in continuous-time dynamical systems. As a consequence of that Class C lasers do satisfy the necessary condition for generating chaos i.e., at least three independent variables are necessary for chaos. However, class B lasers have only two variables and they do not satisfy the condition for generation of chaos. Class B lasers are stable in nature. However, they are easily destabilized by the introduction of external perturbations, resulting in the addition of extra degrees of freedom. Class A lasers are the most stable lasers among the three classes, however, they may show chaotic behaviors by external perturbations with two or more extra degrees of freedom, as for class B lasers.